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Amplitude equations and pattern selection in viscoelastic convection

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Pattern selection and stability in viscoelastic convection are studied in the framework of amplitude equations derived in the vicinity of stationary and oscillatory instabilities. The oscillatory instability corresponds to a Hopf bifurcation with broken translational symmetry. When this instability is the first to appear with increasing Rayleigh number, such systems may be described by coupled one-dimensional complex Ginzburg-Landau equations for counterpropagating waves. The coefficients of these equations, as computed from the underlying Navier-Stokes equations, are such that the selected pattern corresponds to standing waves. The phase dynamics of these waves is derived and leads to coupled Kuramoto-Sivashinsky equations. Their stability range is also determined for different typical fluid parameters. [S1063-651X(96)03108-X]

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I. INTRODUCTION

Convective instabilities of the Rayleigh-Bénard type have been known since the turn of the century, and the properties of the spatial patterns they induce have been extensively studied, either experimentally or theoretically [1]. However, it is only during the last decades that a global understanding of pattern formation in hydrodynamic instabilities has been achieved thanks to both carefully controlled experiments and the development of an appropriate theoretical framework [2]. The related theoretical methods are based on new analytical and numerical tools. The analytical methods are inspired by the mathematics of dynamical systems and the possible reduction of complex dynamics, close to instability points, to much simpler forms, many aspects of which may be studied analytically [3]. This reduction of the dynamics leads to amplitude equations of the Landau-Ginzburg type, which are able to describe the formation, selection, and stability of the convective patterns, quantitatively agreeing with several experimental observations.

In simple Newtonian fluids, the Rayleigh-Bénard instability, which occurs in a fluid layer heated from below, can only lead, at the first instability threshold, to stationary patterns. In this case, the main questions are thus related to the symmetry of the selected structures (rolls or hexagons), to the wavelength selection mechanisms, and to the transition between patterns of different symmetries [4,5].

Of course, in more complex fluids, such as binary mixtures or liquid crystals, oscillatory and even more complex spatiotemporal patterns may also appear near the first convective instability [6,7]. The main properties of these patterns have also been successfully described in the framework of the amplitude equations formalism. Therefore, it appears to be natural to apply this formalism to polymeric fluid convection, where additional degrees of freedom can also lead to a very rich dynamical behavior [8–10].

In fact, the viscoelastic properties of such fluids appear in the constitutive equation, which relates the stress and strain rate tensors. Finding this relation, which should generalize the linear dependence characteristic of Newtonian fluids, is the main purpose of rheology. The simplest constitutive equation capable of describing realistically the viscoelastic properties of diluted polymers, such as Boyer fluids or water solutions of polyacrylamides, is given by the so-called Oldroyd model [11]. In this model, the stress tensor is decomposed into both a polymeric contribution and a solvent contribution. Furthermore, the binary mixture aspects of these fluids are assumed to be irrelevant versus their viscoelastic properties. In the weak shear regime, nonlinear stress-strain rate dependences may be neglected, and the model may be reduced to its linear approximation, known as the Jeffreys model. This model contains three parameters: the static viscosity, the stress relaxation time (which gives an estimate of the time the stress tensor needs to react to a strain change in the system, and is characteristic of a Maxwell fluid), and the retardation time, which results from the fact that the strain dynamics is usually not purely relaxational. The relaxational time is very short in normal fluids but increases strongly in polymeric solutions. The ratio Λ of the retardation and relaxation times is an important rheological parameter, and it varies from, 0 in the case of a Maxwell fluid, to 1 in the case of a Newtonian fluid.

Linear stability analysis has been performed by various authors to determine the onset of convection in such systems, either with free-free or rigid-rigid boundary conditions. It has been found that, besides the usual stationary convection, oscillatory states can also be obtained at onset [12–15]. Which type of convection—stationary or oscillatory—appears first is determined by the values of the rheological parameters. At fixed Prandtl number and relaxation time, it is the stress relaxation time Γ that fixes the relative position of the stationary and oscillatory instability thresholds.

This oscillatory instability can lead to traveling or standing waves, and the stability of these solutions has to be studied in a nonlinear analysis framework beyond the instability threshold. The derivation of amplitude equations for oscillating viscoelastic convection in the weakly nonlinear regime has been presented and analyzed elsewhere [16–21]. Hence,

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the aim of this paper is to review these previous results and to perform the selection and stability analysis of the oscillatory convective patterns, including their phase stability. It will be shown that, while traveling waves are always found unstable, standing waves may be stable in a wide parameter range. In addition, the derivation of the phase dynamics of the standing waves is presented, and the properties of the defects, which may be expected as the result of oscillatory instability associated time and space symmetry breakings, are discussed.

The interest of this study is twofold. On the one hand, it is shown that even small viscoelastic properties may strongly modify the convective patterns. On the other hand, the sensitivity of convective patterns to fluid properties could be used to either determine the rheological parameters of a given fluid or test the soundness of the type of constitutive equation used to describe it.

The paper is organized as follows. In Sec. II, the basic hydrodynamic equations for viscoelastic convection are presented. In Sec. III, the linear stability analysis of the conduction state is performed, and the conditions for the onset of convection are discussed. In Sec. IV, the amplitude equations for the convective patterns are derived and, in Sec. V, the pattern selection and stability, including the phase stability, are analyzed. Finally, conclusions and outlook are presented in Sec. VI.

II. BASIC EQUATIONS AND CONSTITUTIVE EQUATIONS

A layer of incompressible viscoelastic fluid of depth d and of infinite horizontal extent is considered. The fluid is heated from below and remains at rest until a critical temperature gradient is reached. In the Boussinesq approximation, the balance equations can be written as

$$\nabla \cdot \mathbf{v} = 0, \tag{1}$$

$$\rho_0[\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}] = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \rho_0[1 - \alpha(T - T_0)\mathbf{g}], \tag{2}$$

$$[\partial_t T + (\mathbf{v} \cdot \nabla)T] = \kappa \nabla^2 T, \tag{3}$$

where $\mathbf{v}$ is the velocity field, p the pressure, τ the extra stress tensor, T the temperature, $\mathbf{g}$ the acceleration of gravity, α the thermal expansion coefficient, T_0 the temperature, ρ_0 the reference density, and κ the thermal diffusivity.

In a normal incompressible fluid, the extra stress tensor is related to the strain tensor via the Newton law $\boldsymbol{\tau} = 2\eta\mathbf{D}$, where $\mathbf{D}[v] = [(\nabla \cdot \mathbf{v}) + (\nabla \cdot \mathbf{v})^T]/2$ is the strain rate tensor, η is the viscosity and a superscript T stands for transposition. But, usually, a more general constitutive relation between stress and strain rate $\boldsymbol{\tau} = \boldsymbol{\tau}(\mathbf{D})$ is necessary to describe the behavior of complex polymeric fluids. This relation is subjected to symmetry restrictions. A type of constitutive relation that satisfies these restrictions and that may be further justified by the kinetic theory of dumbbells are those proposed by Oldroyd. These models, developed in the 1950s, include particular cases that are widely used for different kinds of polymeric solutions.

In the Oldroyd model, the constitutive equation is written as

$$\boldsymbol{\tau} + \lambda_1 D_t \boldsymbol{\tau} = 2\eta_0[\mathbf{D} + \lambda_2 D_t \mathbf{D}]. \tag{4}$$

This equation contains three parameters: η_0, the static (or zero shear viscosity); λ_1, the so-called relaxation time; and λ_2, the retardation time $(0 \leqslant \lambda_2 \leqslant \lambda_1)$. In rheology, the so-called frame-indifferent principle should be applied. It states that the constitutive equations must be invariant under local rigid rotations. Then, in order to satisfy the frame invariance, objective time derivatives ought to be taken in the constitutive equation. The symbol D_t is denoted as an invariant (frame-indifferent) time derivative, defined as

$$D_t \boldsymbol{\tau} = \partial_t \boldsymbol{\tau} + (\mathbf{v} \cdot \nabla)\boldsymbol{\tau} + \boldsymbol{\tau} \cdot \mathbf{W} - \mathbf{W} \cdot \boldsymbol{\tau} + a[\mathbf{D} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \mathbf{D}], \tag{5}$$

where $\mathbf{W}$ is the skew-symmetric part of the velocity gradient, and a is a phenomenological parameter that can vary between -1 and 1. The case $a = -1$ corresponds to the lower convected Jeffreys model (Oldroyd B), $a = 0$ to the so-called corotational Jeffreys model, and $a = 1$ to the upper convected Jeffreys model (Oldroyd A).

The constitutive equation can be derived from a molecular theory, in which the polymer molecules are considered as noninteracting Hookean elastic dumbbells immersed in a Newtonian solvent. The stress tensor $\boldsymbol{\tau}$ can then be decomposed as

$$\boldsymbol{\tau} = \boldsymbol{\tau}_s + \boldsymbol{\tau}_p, \tag{6}$$

where $\boldsymbol{\tau}_s$ is the Newtonian solvent contribution

$$\boldsymbol{\tau}_s = 2\mu_s \mathbf{D}, \tag{7}$$

and the polymeric contribution of a concentrated Maxwellian convected model is given by the contribution of the solvent and polymer viscosities η_s and η_p, respectively. Moreover, the relaxation and retardation times are related by

$$\lambda_2 = \frac{\eta_s}{\eta_s + \eta_p} \lambda_1. \tag{8}$$

For weak shears, the linear approximation of the Oldroyd model leads to the more simple Jeffreys constitutive equation for a polymeric fluid

$$(1 + \lambda_1 \partial_t)\boldsymbol{\tau} = 2\eta_0(1 + \lambda_2 \partial_t)\mathbf{D}. \tag{9}$$

The relaxation time λ_1 gives an idea of the time the stress tensor needs to react to a strain change in the system. Usually, this time is very small in normal fluids ($\lambda_1 = 10^{-12}$ s for water) but can be sufficiently large in polymeric solutions. However, the evolution of strain and stress often needs a second time because the strain dynamics is not purely relaxational: there is an interrelation between stress and strain relaxation. As an example, in the polymeric solution Boger fluid (B11), constituted by water (1.6%), syrup (98.3%), and polyacrylamide (0.1%). The relaxation and retardation times are $\lambda_1 = 2.54$ s, and $\lambda_2 = 1.97$ s [22], respectively.

The quiescent solution of the system of equations (1)–(4) is simply $\mathbf{v} = 0$, $\boldsymbol{\tau} = 0$, $T = T_1 - [(T_1 - T_2)/d]z$, where T_1 and T_2 are the temperatures in the hot lower and cold upper plates, respectively. This corresponds to a purely conductive state. Now, as usual, the stability of this reference state is studied via the evaluation of small perturbations or linear stability analysis. This analysis is performed by rescaling the variables by d (length), d^2/κ (time), κ/d (velocity), $\eta_0\kappa/d^2$ (stress tensor), and $\Delta T=(T_1-T_2)$ (temperature).

For the sake of simplicity, the analysis was limited to two-dimensional motion. Therefore, we can use a stream function $\psi(x,z,t)$ that gives the velocity field $(v_x,v_z)=(\partial_z\psi,-\partial_x\psi)$. Due to its symmetry properties, the extra stress tensor has only three independent coefficients: τ_{xx}, τ_{zz}, $\tau_{zx}=\tau_{xz}$. Instead of these individual components, the following three scalar quantities are usually considered in rheology: the trace $U=\tau_{xx}+\tau_{zz}$, the normal stress difference $S=\tau_{xx}-\tau_{zz}$, and the in-plane shear stress τ_{xz}. In rescaled variables, Eqs. (1)–(4) reduce to a system of evolution equations for the five scalar perturbation ψ, θ, τ_{xz}, S, and U. (Here θ denotes the nondimensional temperature perturbation.)

The set of adimensional perturbation equations of the conduction state is written in its compact form

$$\partial_t\mathbf{L}_r\boldsymbol{\phi}=\mathbf{M}_\rho\boldsymbol{\phi}+\mathbf{N}_r(\boldsymbol{\phi},\boldsymbol{\phi}), \tag{10}$$

where the vector $\boldsymbol{\phi}$ is simply $\phi(x,z,t)=[\psi,\theta,\tau_{xz},S,U]^T$, while r represents the set of external parameters $[R,P,\Gamma,\Lambda]$ where $R=\rho_0\alpha\Delta T d^3/\eta_0\kappa$ is the Rayleigh number, $P=\eta/\rho_0\kappa$ is the Prandtl number, $\Gamma=\lambda_1\kappa/d^2$ is the nondimensional stress relaxation time, and $\Lambda=\lambda_2/\lambda_1$ is the ratio between retardation and relaxation times. **L** and **M** are defined as

$$\mathbf{L}=\begin{bmatrix} \nabla^2 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ -\Gamma\Lambda\Delta^2 & 0 & \Gamma & 0 & 0\\ -4\Gamma\Lambda\partial^2_{xz} & 0 & 0 & \Gamma & 0\\ 0 & 0 & 0 & 0 & \Gamma \end{bmatrix}, \tag{11}$$

$$\mathbf{M}=\begin{bmatrix} 0 & -PR\partial_x & P\Delta^2 & P\partial^2_{xz} & 0\\ -\partial_x & \nabla^2 & 0 & 0 & 0\\ \Delta^2 & 0 & -1 & 0 & 0\\ 4\partial^2_{xz} & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \tag{12}$$

while $N(\boldsymbol{\phi},\boldsymbol{\phi})$ is the nonlinear part of the dynamics and takes the following form:

$$\mathbf{N}(\boldsymbol{\phi},\boldsymbol{\phi})=\begin{bmatrix} -J(\psi,\nabla^2\psi)\\ -J(\psi,\theta)\\ \Gamma\Lambda\{J(\psi,\Delta^2\psi)+2(\partial^2_{xz}\psi)(\nabla^2\psi)\}-\Gamma\{J(\psi,\tau_{xz})-\frac{1}{2}(\nabla^2\psi)S+\frac{1}{2}a(\Delta^2\psi)U\}\\ \Gamma\Lambda\{4J(\psi,\partial^2_{xz}\psi)-2(\nabla^2\psi)(\Delta^2\psi)\}+\Gamma\{2(\nabla^2\psi)\tau_{xz}-J(\psi,S)+2a(\partial^2_{xz}\psi)U\}\\ -2\Gamma\Lambda a\{(2\partial^2_{xz}\psi)^2+(\Delta^2\psi)^2\}+\Gamma\{2a(\Delta^2\psi)\tau_{xz}+2a(\partial^2_{xz}\psi)S-J(\psi,U)\} \end{bmatrix}. \tag{13}$$

$J(f,g)=\partial_z f\partial_x g-\partial_x f\partial_z g$ denotes the Jacobian, and $\Delta^2=\partial^2_{zz}-\partial^2_{xx}$, $\nabla^2=\partial^2_{xx}+\partial^2_{zz}$.

This system of equations must be supplemented with appropriate boundary conditions (bc). The simplest choice is an extension of the conducting, stress-free bc, that can be written as

$$\psi=\partial_z^2\psi=0 \quad \text{at } z=0,1 \tag{14a}$$

and implies

$$\tau_{xz}=\partial_z S=U=0 \quad \text{at } z=0,1. \tag{14b}$$

The second (thermal) bc is equivalent to

$$\theta=0. \tag{14c}$$

Although unrealistic, these bc have the advantage of providing the analytic resolution of the instability problems. Furthermore, as shown in [17], the use of realistic bc in the linear stability analysis does not affect the results quantitatively, but it only leads to quantitative changes in the instability threshold.

III. LINEAR STABILITY

In considering small perturbations, ϕ, the nonlinear term in Eq. (10), may be neglected and, therefore, the corresponding linear stability problem reduces to the equation

$$\partial_t\mathbf{L}_r\boldsymbol{\phi}=\mathbf{M}_r\boldsymbol{\phi}, \tag{15}$$

with the bc Eq. (14). Notice that in the linear approximation the general Oldroyd model reduces to the Jeffreys model. Looking at the linear operators, L_r and M_r Eqs. (11) and (12), it can be observed that the variable U is decoupled from the other variables, and it has a purely decaying exponential evolution. Hence, this variable is always linearly stable and needs not to be considered in the linear stability analysis. As usual, the solutions of the system (15) with the bc (14) may be expressed in the form of the normal modes,

$$\phi(x,z,t)=e^{st}\tilde{\psi}(x,z), \tag{16}$$

where the vector $\boldsymbol{\phi}$ has only four component vectors after the elimination of U and, therefore, the following expression can be taken for the modes:

$$\tilde{\psi}(x,z)=\begin{bmatrix} A\ \sin(mkx)\sin(n\pi z)\\ B\ \cos(mkx)\sin(n\pi z)\\ C\ \sin(mkx)\sin(n\pi z)\\ D\ \cos(mkx)\cos(n\pi z) \end{bmatrix}, \tag{17}$$

where A, B, C, and D are arbitrary constants, and $s=\sigma+i\omega$ is in general a complex quantity (σ is the growth factor of a perturbation and ω its frequency). Introducing Eqs. (16) and (17) in Eq. (15), the linear problem is reduced to the eigenvalue problem,

$$sL\tilde{\psi}(x,z)=M\tilde{\psi}(x,z). \qquad (18)$$

This equation has nontrivial solutions if the following condition is satisfied:

$$\det(M-sL)=s^j+\sum_{i=0}^{j-1}a_is^i=0, \quad m,n\geqslant 1. \qquad (19)$$

It can be proved that the critical modes (as in the Rayleigh-Bénard problem in a normal fluid) correspond to the case $m=n=1$. In general, for m and n different from 1, the characteristic equation (19) is quartic. However, in the particular case $m=n=1$, one root is always real negative $s=-1/\Gamma$ and, therefore, it corresponds to a stable eigenvector. Hence, the characteristic Eq. (19) can be reduced to the cubic equation

$$P(s)=s^3+a_2s^2+a_1s+a_0=0, \qquad (20)$$

where

$$a_2=\frac{1}{\Gamma}+q^2(1+\Lambda P), \qquad (21)$$

$$a_1=\frac{Pk^2}{q^2}\left(\frac{q^6\Lambda}{k^2}+\frac{q^4}{k^2P\Gamma}(1+P)-R\right), \qquad (22)$$

$$a_0=\frac{Pk^2}{\Gamma q^2}\left(\frac{q^6}{k^2}-R\right) \qquad (23)$$

and $q^2=k^2+\pi^2$. The roots of the polynomial (20) give the different bifurcations that may appear in this problem. We will study them in some detail in the following subsections.

A. Stationary instability

This instability corresponds to a simple zero eigenvalue $(s=0)$, which is obtained when $a_0=0$ and $a_1>0$. The first condition gives the marginal stability curve

$$a_0=0\rightarrow R_s=\frac{q^6}{k^2}, \qquad (24)$$

which is the same as in normal fluids, because viscoelastic properties do not affect the stationary case. The minimum value of this curve gives the instability threshold defined by the critical Rayleigh and wave numbers $R_{cs}=27\pi^4/4$ and $k_{cs}=\pi/\sqrt{2}$, respectively. (The subscript s indicates a stationary solution, while subscript c indicates critical values of the corresponding instability.) The condition $a_1>0$ corresponds to values of Γ that respond to the relation

$$\Gamma<\frac{1+P^{-1}}{q^2(1-\Lambda)}. \qquad (25)$$

B. Oscillatory instability

When $a_0=a_1a_2$, and $a_1>0$ is a pair of complex conjugated imaginary eigenvalues, $s=\pm i\omega$ may be the solution to the characteristic polynomial (20), and this corresponds to an oscillatory instability. The first condition defines the corresponding marginal curve

$$a_0=a_1a_2\rightarrow R_0=\frac{q^6\Lambda}{k^2}+\frac{q^2}{k^2\Gamma(1+\Lambda P)}\times\left(q^2(2\Lambda+\Lambda P+P^{-1})+\frac{1+P}{\Gamma P}\right). \qquad (26)$$

Examples of these curves can be seen in Refs. [14,15,17]. In general, the critical values R_{co} and k_{co} (the subscript o stands for oscillatory instability) must be obtained numerically from Eq. (26).

The second condition gives the oscillation frequency ω as a function of the physical parameters

$$a_1>0\rightarrow\omega^2=\frac{q^2P\Gamma(1-\Lambda)-(1+P)}{\Gamma^2(1+\Lambda P)}>0\rightarrow\Gamma>\frac{1+P^{-1}}{q^2(1-\Lambda)}. \qquad (27)$$

Note that the oscillatory instability can appear before or after the stationary instability, according to Eqs. (25) and (27).

This problem has been solved by different authors [11–14]. The Rayleigh number for the oscillatory instability for different nondimensional stress relaxation times Γ is shown as a function of the wave number k (Fig. 1), for a diluted polymer ($P=100.0$ and $\Lambda=0.75$). The dotted line represents the stationary convection.

In Fig. 2, the critical Rayleigh number R_c, the critical wave number k_c, and frequency ω_c for the onset of overstability are plotted as functions of the logarithm of the nondimensional stress relaxation time Γ for both a diluted polymeric fluid ($P=100.0$, $\Lambda=0.75$) and a Maxwellian fluid ($P=1000.0$, $\Lambda=0.0$). The dashed lines indicate the stationary instabilities.

IV. WEAKLY NONLINEAR ANALYSIS AND AMPLITUDE EQUATIONS

A nonlinear analysis must be carried out in order to determine the type of convective motion that could be developed beyond the instability threshold. Close to the instability, this analysis can be made in the framework of a weakly nonlinear analysis, which leads to the derivation of amplitude equation of the Ginzburg-Landau type [3]. This approach basically assumes that the constant amplitudes of the linear analysis are now space and time dependent above the convection threshold. These amplitudes evolve on the space and time scales of the unstable modes.

The solutions of the nonlinear problem in the neighborhood of the convection threshold, i.e., for small values of the reduced Rayleigh number $\varepsilon=(R-R_c)/R_c$, and small amplitudes, may be expressed as follows: (1) for stationary convection,

$$\boldsymbol{\phi}_0(x,z,t)=\mathbf{U}_{os}(z)[A\ \exp(ik_{cs}x)+A^*\ \exp(-ik_{cs}x)] \qquad (28a)$$

and (2) for oscillatory convection,

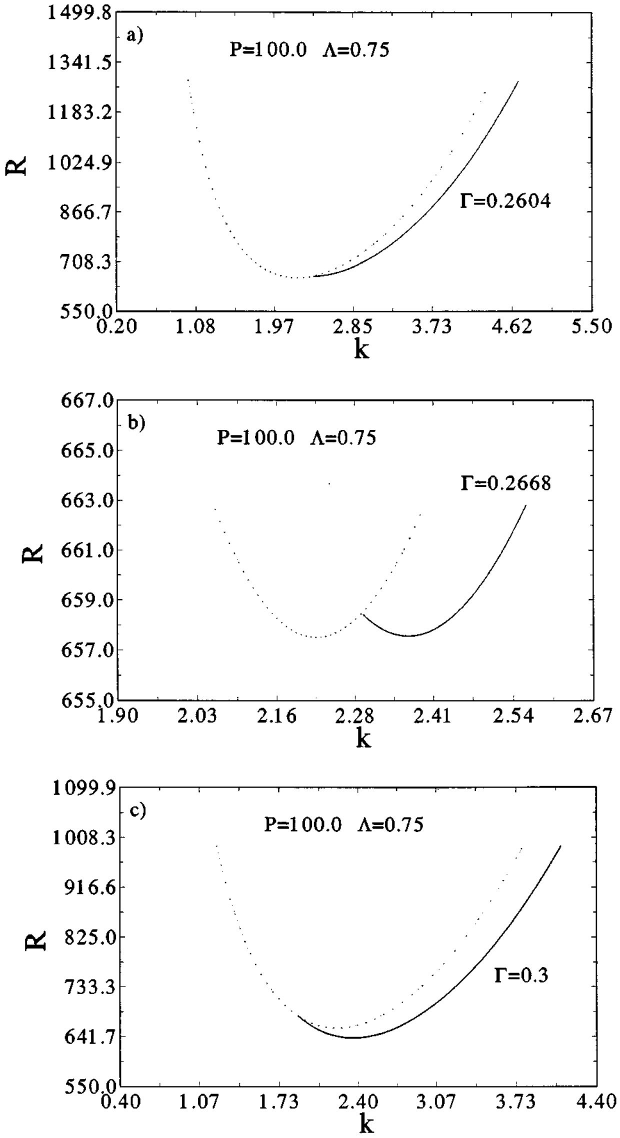

FIG. 1. The Rayleigh number for the oscillatory instability for different nondimensional stress relaxation times Γ is shown as a function of the wave number k, for a diluted polymer ($P=100.0$ and $\Lambda=0.75$). The dotted line represents the stationary convection.

$$\boldsymbol{\phi}_0(x,z,t)=\mathbf{U}_{oo}(z)\{A\ \exp[i(k_{co}x+\omega_c t)] + B\ \exp[-i(k_{co}x-\omega_c t)]\}+\text{c.c.}, \quad (28b)$$

where c.c. refers to the complex conjugate of the first term. The perturbation vector $\boldsymbol{\phi}_0$ gives rise to traveling waves (TW) for $\omega_c \neq 0$ either $A=0$ or $B=0$, and stationary waves (SW) $\omega_c \neq 0$ if $|A|=|B|$.

The purpose of this paper is to derive the evolution equation for the amplitudes A and B of Eq. (28), i.e., the amplitude equations, and to compute both the nonlinear coefficients that determine the character of the bifurcation (subcritical and supercritical) and the type of convective pattern that may be obtained in the system.

A formal scheme for the derivation of the amplitude equations following Refs. [2] and [3] is described below. First, we write Eq. (10) in its compact form,

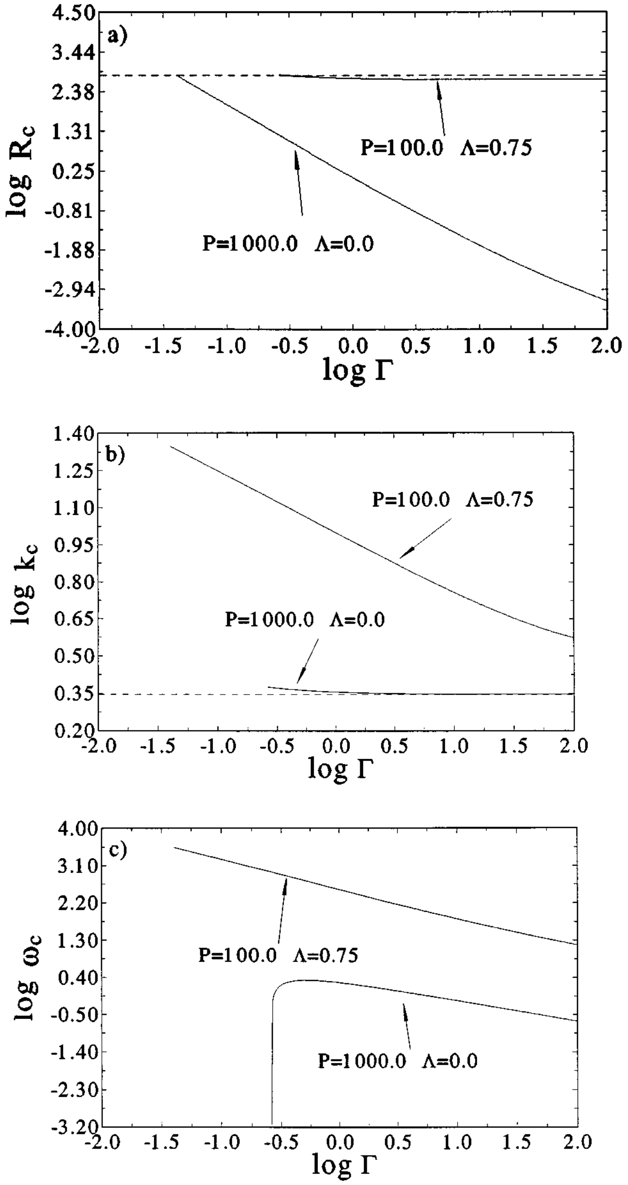

FIG. 2. The critical Rayleigh number R_c, the critical wave number k_c, and frequency ω_c for the onset of overstability are plotted as functions of the logarithm of the nondimensional stress relaxation time Γ for a diluted polymeric fluid ($P=100.0$, $\Lambda=0.75$) and a Maxwellian fluid ($P=1000.0$, $\Lambda=0.0$). The dashed lines indicate the stationary instabilities. In all figures the logarithm is plotted to the base 10.

$$\mathbf{L}\boldsymbol{\phi}=\mathbf{N}(\boldsymbol{\phi},\boldsymbol{\phi}), \quad (29a)$$

where

$$\mathcal{L}=\partial_t\mathbf{L}-\mathbf{M}. \quad (29b)$$

Amplitudes A and B vary on the slow scales

$$X=\varepsilon^{1/2}x, \quad T_1=\varepsilon^{1/2}t, \quad T=\varepsilon t, \quad (30)$$

which are treated as independent variables. Hence,

$$A=A(X,T_1,T,), \quad B=B(X,T_1,T) \quad (31)$$

is obtained. We expand the solution of (29) with respect to $\varepsilon^{1/2}$,

$$\phi(x,z,t,X,T_1,T)=\varepsilon^{1/2}\phi_0+\varepsilon\phi_1+\varepsilon^{3/2}\phi_2+\cdots\ , \tag{32}$$

where the arguments on the left-hand side are repeated on the right-hand side, and substitute $\partial_t \to \partial_t+\varepsilon^{1/2}\partial_{T_1}+\varepsilon\partial_T$, $\partial_x \to \partial_x+\varepsilon^{1/2}\partial_x$, according to Eq. (30) as well as $R=R_c(1+\varepsilon)$. Inserting all this into Eq. (29),

$$\mathcal{L}_0\phi_0=0, \tag{33}$$

$$\mathcal{L}_0\phi_1=\mathbf{N}(\phi_0,\phi_0)-\mathcal{L}_1\phi_0, \tag{34}$$

$$\mathcal{L}_0\phi_2=+\mathbf{N}_2(\phi_0,\phi_1)+\mathbf{N}(\phi_1,\phi_0)-\mathcal{L}_1\phi_1-\mathcal{L}_2\phi_2 \tag{35}$$

yields at successive orders of $\varepsilon^{1/2}$.

The new operators $\mathcal{L}_1$,$\mathcal{L}_2$ and $\mathbf{N}_1$,$\mathbf{N}_2$ are complicated functions of the new variables, but their explicit form may remain unknown, as is shown below. $\mathcal{L}_0$ is the linear operator $\mathcal{L}$, defined in Eq. (11), and R is replaced by R_c; then, Eq. (33) is simply the linear part of Eq. (29) presenting the solution (28) already known, but where the amplitudes are now functions of the slow variables [see Eq. (31)]. The right-hand side (rhs) of Eq. (34) depends only on ϕ_0, which is already calculated at order $\varepsilon^{1/2}$ and is found in the form of Eq. (28). Therefore, Eq. (34) is an inhomogeneous boundary-value problem for $\phi(x,z,t,X,T_1,T)$ that can be solved by integration. After inserting ϕ_0 and ϕ_1 into Eq. (35), there is no real need to solve this equation. Instead, by projecting the whole equation onto $\phi_0^\dagger$, where $\phi_0^\dagger$ is the solution to the adjoint equation of (33), the rhs of Eq. (35) yields a solvability condition. By now rescaling the old variables x and t [and by choosing a slightly different form of the expansion (32), where the factor $\varepsilon^{1/2}$ is included in ϕ_0], these solvability conditions are the amplitude equations for A and B.

A. Amplitude equation for the stationary instability

In many other problems of this type in the regime where the first instability is the stationary one, the amplitude equation is

$$\partial_t A=\mu A+\alpha\partial_x^2 A-\beta A^3, \tag{36}$$

where coefficients μ, α, and β are functions of parameters a, P, Γ, Λ, k_{cs}, and R_{cs}. The bifurcation is always supercritical for $a=\pm 1$, since β is strictly real and positive. We found that this is also true in the particular case of Maxwell fluid ($\Lambda=0.0$), which is opposed to what some authors have found that by using simple material derivatives instead of objective material derivatives, supercritical and subcritical transitions become separated by a tricritical point. In Fig. 3(a), β is represented as a function of Γ for a different Maxwell fluid, and upper and lower convective Oldroyd fluids ($a=\pm 1$).

It must be emphasized that the value of a ($|a|<0.825$) is crucial to the existence of subcritical bifurcation and polycritical points observed, for instance, on the corotational Jeffreys model ($a=0$) [Fig. 3(b)]. As a function of Γ, coefficient β runs from positive values for viscoelastic fluids, whose concentrations range from the Maxwell ($\Lambda=0.0$) to a high diluted polymeric fluid ($\Lambda=0.75$).

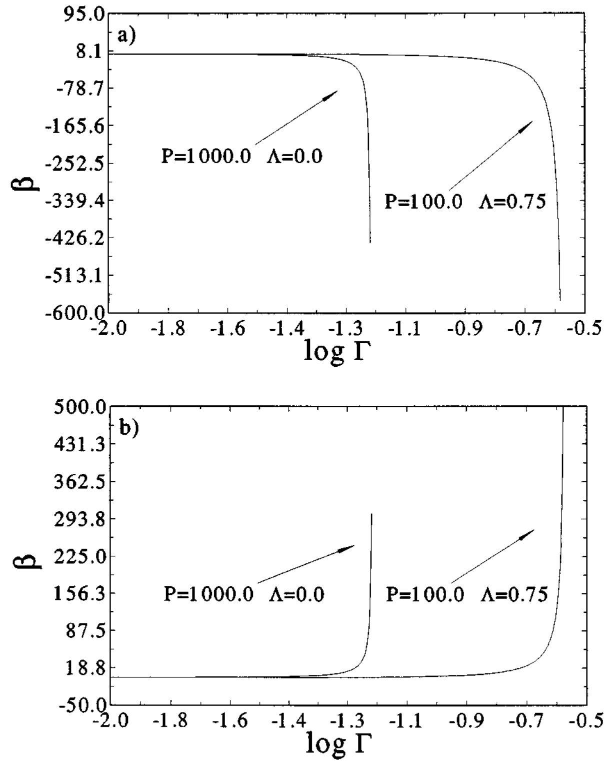

FIG. 3. The cubic coefficient in the stationary amplitude β is represented as a function of Γ for both a concentrated Maxwellian fluid ($P=1000.0$, $\Lambda=0.0$) and a diluted polymeric fluid ($P=1000.0$, $\Lambda=0.75$): (a) for upper and lower convective Oldroyd fluids ($a=\pm 1$); (b) for corrotational Jeffreys model ($a=0$).

B. Amplitude equation for oscillatory convection

As in binary fluids, oscillatory convection is possible when two time scales compete in the system. Since the instability occurs at a finite wave number, traveling or standing waves may be expected beyond threshold (left and right traveling waves correspond to $A=0$ or $B=0$, and standing waves correspond to $|A|=|B|$).

In applying the formalism presented in Eq. (28), the nonlinear spatial behavior of the system may be described, in this case, by the following coupled Ginzburg-Landau equations:

$$\begin{aligned}(\partial_t+v\partial_x)A&=\mu A+(\alpha_r+i\alpha_i)\partial_x^2 A-(\beta_r+i\beta_i)|A|^2A\\&\quad-(\gamma_r+i\gamma_i)|B|^2A,\\(\partial_t-v\partial_x)B&=\mu B+(\alpha_r+i\alpha_i)\partial_x^2 B-(\beta_r+i\beta_i)|B|^2B\\&\quad-(\gamma+i\gamma_i)|A|^2B,\end{aligned} \tag{37}$$

where the sets of linear and nonlinear coefficients are $(v,\mu,\alpha_r,\alpha_t)$ and $(\beta_r,\beta_i,\gamma_r,\gamma_i)$, respectively, v represents the group velocity, μ measures the deviation from the instability, α_r is a diffusive term, α_i represents dispersive effects, β_i and γ_i are nonlinear renormalization frequencies, and β_r and γ_r are nonlinear saturation terms.

The various nonlinear coefficients β_r and γ_r are presented as functions of the nondimensional relaxation time Γ for both concentrated Maxwellian fluid ($P=1000.0$, $\Lambda=0.0$) and a

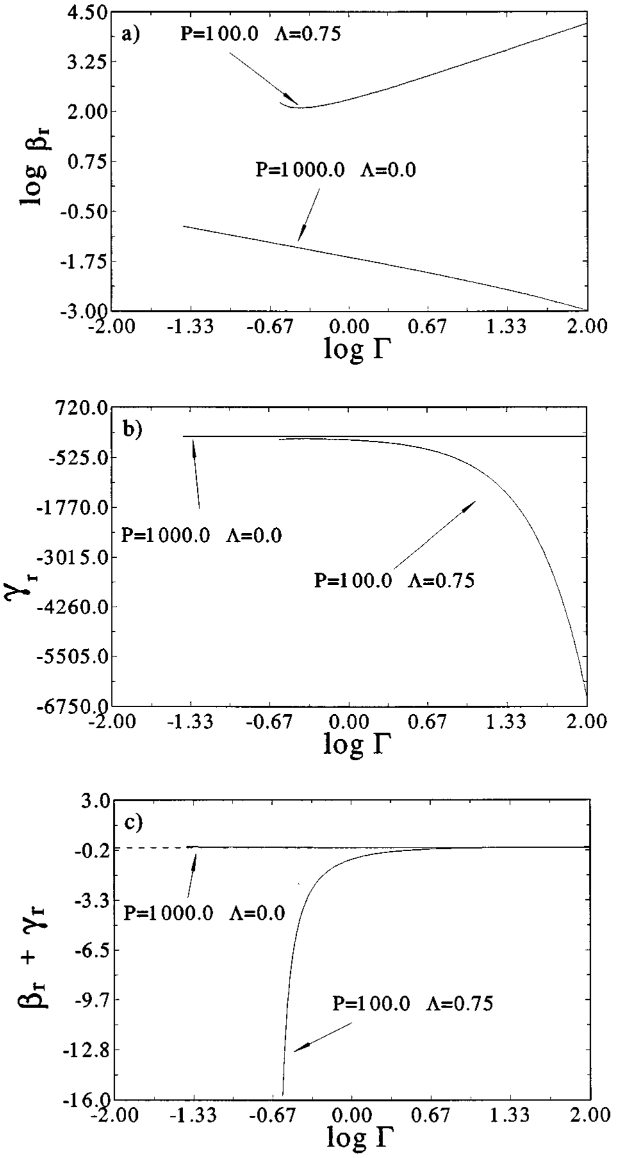

FIG. 4. The nonlinear coefficients β_r, γ_r, and β_{SW} are presented as functions of the nondimensional relaxation time Γ for both a concentrated Maxwellian fluid ($P=1000.0$, $\Lambda=0.0$) and a diluted polymeric fluid ($P=1000.0$, $\Lambda=0.75$) for upper and lower convective Oldroyd fluids ($a=\pm 1$).

diluted polymeric fluid ($P=1000.0$, $\Lambda=0.75$) [Figs. 4(a) and 4(b)]. As coefficient β_r is always positive, we could expect a supercritical bifurcation towards TW. However, since γ_r appears to be negative for both fluids, the traveling waves should be unstable, as is shown for upper and lower convective Oldroyd fluids ($a=\pm 1.0$) in the next section.

Effectively, since β_1 is positive, the uniform solutions of Eq. (37) may be either (1) of the traveling wave type,

$$|A|=\sqrt{\frac{\mu}{\beta_r}}\exp\left(\frac{\beta_i\mu}{\beta_r}t\right), \quad |B|=0 \quad \text{for } \mu>0 \tag{38}$$

or

$$|A|=0, \quad |B|=\sqrt{\frac{\mu}{\beta_r}}\exp\left(\frac{\beta_i\mu}{\beta_r}t\right), \tag{39}$$

or (2) of the standing wave type,

$$|A|=|B|=\sqrt{\frac{\mu}{\beta_r+\gamma_r}}\exp\left(\frac{\beta_i+\gamma_i}{\beta_r+\gamma_r}\mu t\right). \tag{40}$$

Evidently, this supercritical bifurcation towards SW implies that $\beta_r+\gamma_r>0$, which is true for high concentrated fluids, mainly when $\Gamma<64$. For $\Gamma<64$, the stability is provided by the fact that γ_r is negative.

As for the standing waves, the phase stability determining the complete stability range in the supercritical regime should be studied. While the amplitude stability of the curves is determined by $\gamma_r<0$, complete stability needs to be determined by a phase stability analysis.

The analysis for standing waves [Fig. 4(c)] shows that the corresponding coefficient β_{SW} is given by the addition of γ, and the previous coefficient β_{TW} is evaluated for the traveling waves. The value of $\beta_{\mathrm{SW}}=\gamma_r+\beta_{\mathrm{TW}}$ is always negative for diluted polymers. Nevertheless, it is positive for highly concentrated fluids, when Γ is lower than 64, a range where there are supercritical solutions for standing waves. A tricritical bifurcation is found at this point, and since for higher values of Γ the coefficient becomes negative, the bifurcations are subcritical. It is necessary then to apply a fifth-order nonlinear analysis of the amplitude equation in order to describe the evolution of the physical system.

The situation is completely different for corotational Jeffreys model ($a=0$). Results based on this model are shown in Figs. 5(a)–5(c). These results show that the convective properties can change drastically, depending on the kind of derivative used in the constitutive equation.

V. PATTERN SELECTION AND STABILITY

When the stationary instability is the first instability to appear, in increasing the Rayleigh number, the system will develop roll patterns corresponding to the steady solutions of the amplitude equation (36). These solutions form the family

$$A_K=\sqrt{\frac{\mu-\alpha k^2}{\beta}}e^{i[kx+\varphi]}, \tag{41}$$

where φ is an arbitrary phase.

These solutions exist in the range $-\sqrt{\mu/\alpha}\leqslant k\leqslant\sqrt{\mu/\alpha}$ ($\mu\geqslant 0$), but their stability range is reduced, as usual, by the Ekhaus instability, and is thus given by

$$-\sqrt{\frac{\mu}{3\alpha}}\leqslant k\leqslant\sqrt{\frac{\mu}{3\alpha}}. \tag{42}$$

On the other hand, when the oscillatory instability is the first to appear, the possible patterns will correspond to the solutions of the amplitude equations (37). These solutions may be either of the traveling or the standing wave type. In order to analyze the amplitude equations with standard methods, these equations could be written as

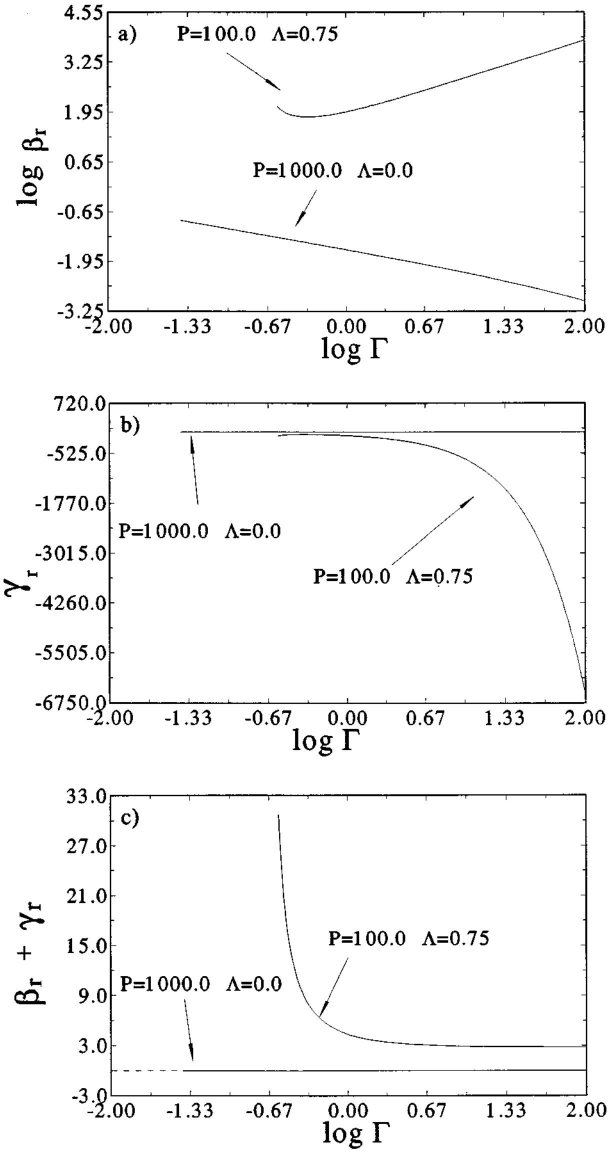

FIG. 5. The nonlinear coefficients β_r, γ_r, and β_{SW} are presented as functions of the nondimensional relaxation time Γ for both a concentrated Maxwellian fluid ($P=1000.0$, $\Lambda=0.0$) and a diluted polymeric fluid ($P=1000.0$, $\Lambda=0.75$) for the corotational Jeffreys model ($a=0$).

$$\partial_t A + c\partial_x A = \mu A + (1+i\alpha)\partial_x^2 A - (1+i\beta)|A|^2 A - (\gamma + i\delta)|B|^2 A,$$
$$\partial_t B - c\partial_x B = \mu B + (1+i\alpha)\partial_x^2 B - (1+i\beta)|B|^2 B - (\gamma + i\delta)|A|^2 B, \tag{43}$$

with $\alpha=\alpha_i/\alpha_r$, $\beta=\beta_i/\beta_r$, $\gamma=\gamma_r/\beta_r$, $\delta=\gamma_i/\beta_r$, and $c=v/\sqrt{\alpha_r}$. Equations (43) admit two families of traveling wave solutions given by

$$A_k=\sqrt{\mu-k^2}\exp i\{k(x-ct)-\beta\mu t-(\alpha-\beta)k^2t\}, \quad B_k=0, \tag{44}$$

or

$$A_k=0, \quad B_k=\sqrt{\mu-k^2}\exp i\{-k(x+ct)-\beta\mu t - (\alpha-\beta)k^2t\}. \tag{45}$$

The linear stability of such traveling waves versus standing waves may be easily analyzed. In fact, the linear evolution of the Fourier transform of the wave amplitude B, where subindex q represents the Fourier transform in the presence of a traveling wave A_k, is given by

$$(\partial_t - iqc)B_q = \mu B_q - q^2(1+i\alpha)B_q - (\gamma+i\delta)B_q|A_k|^2. \tag{46}$$

Hence, the maximum linear growth rate of these modes, corresponding to the $q=0$ mode, is

$$R_e\omega = \mu - \gamma(\mu - k^2). \tag{47}$$

As a result, since γ is found to be negative in this problem, the traveling waves are always unstable versus the standing waves.

The family of standing wave solutions of Eq. (43), which is given by

$$A_k = Q \exp i[\Omega_A t + kx], \tag{48}$$

$$B_k = Q \exp i[\Omega_A t - kx], \tag{49}$$

with

$$Q=\sqrt{\frac{\mu-k^2}{1+\gamma}}, \tag{50}$$

$$\Omega_A = -kc - k^2\alpha - \frac{(\beta+\delta)(\mu-k^2)}{(1+\gamma)}, \tag{51}$$

$$\Omega_B = +kc - k^2\alpha - \frac{(\beta+\delta)(\mu-k^2)}{(1+\gamma)}, \tag{52}$$

should be considered.

These standing waves appear via a supercritical bifurcation at $\mu=0$ when $\beta_{\mathrm{SW}}=1+\gamma>0$, which is true for highly concentrated polymers, namely, when Γ is lower than 64, as may be seen in Fig. 4(c). For diluted polymers, β_{SW} is always negative, and an inverted bifurcation is expected. The analysis of this situation would require going further, that is, to the fifth-order nonlinearities in the amplitude equation.

The situation is completely different for the corotational Jeffreys model (when $a=0$). The values of the parameters for this model are shown in Fig. 5. In this case, β_{SW} is always positive for a diluted polymer and negative for a concentrated polymer.

These results clearly show that the nature of the bifurcation and of the convective patterns can change drastically, depending on the exact nature of the constitutive equation. Therefore reliable experimental results would be highly desirable in solving this problem in order to test the validity of the constitutive relations used to describe the viscoelastic properties of the fluids under consideration.

Due to the complexity of the computation required to obtain the fifth-order terms of the amplitude equations, this paper will be limited to the case $\beta_{\mathrm{SW}}>0$, where a supercritical bifurcation towards standing waves may be expected.

Waves with wave number q_c+k may thus exist in the range defined by $k^2<\mu$. Since γ is negative, the waves have stable amplitudes. In effect, the linear evolution equations for amplitude perturbations (keeping the phase fixed in this way) are given in the Fourier transform by

$$\partial_t a_k=-(2Q^2+k^2)a_k-2\gamma Q^2 b_k, \tag{53}$$

$$\partial_t b_k=-2\gamma Q^2 b_k-(2Q^2+k^2)b_k, \tag{54}$$

$$|A_k|=Q+a_k, \quad |B_k|=Q+b_k. \tag{55}$$

The fact that $1+\gamma>0$ and $\gamma<0$ ensures the decay of these perturbations. Thus, the stability domain will finally be determined by the phase stability. In order to perform this analysis, the phase dynamics of these solutions will have to be determined.

Due to the symmetry-breaking properties of the oscillatory instability, the phases of the waves are marginal modes evolving on the longest time and space scales of the system. As a result, they govern the asymptotic evolution of the system, and their dynamics may be obtained via the adiabatic elimination of the wave amplitudes.

The following expressions,

$$A=(Q+a)\exp i\{\Omega_A t+kx+\varphi_A\}, \tag{56}$$

$$B=(Q+b)\exp i\{\Omega_B t-kx+\varphi_B\}, \tag{57}$$

will be substituted in the amplitude equations (43). The amplitude perturbations a and b may be adiabatically eliminated through the standard procedure, and after some algebra, we have the following phase equations for the Fourier modes of the phases φ_A and φ_B,

$$\partial_t\begin{pmatrix}(\varphi_A+\varphi_B)_q\\(\varphi_A-\varphi_B)_q\end{pmatrix}=\mathbf{L}^*\begin{pmatrix}(\varphi_A+\varphi_B)_q\\(\varphi_A-\varphi_B)_q\end{pmatrix}+\mathbf{N}^*, \tag{58}$$

where the linear $\mathbf{L}^*$ and the nonlinear $\mathbf{N}^*$ matrixes are

$$\mathbf{L}^*=\begin{pmatrix}-D_+ & -E_+\\-E_- & -D_-\end{pmatrix} \tag{59}$$

and

$$\mathbf{N}^*=\begin{pmatrix}-F_+\{[\partial_x(\varphi_A+\varphi_B)]_q^2+[\partial_x(\varphi_A-\varphi_B)]_q^2\}-G_+\{\partial_x(\varphi_A-\varphi_B)\partial_x(\varphi_A+\varphi_B)\}_q\\-G_-\{[\partial_x(\varphi_A+\varphi_B)]_q^2+[\partial_x(\varphi_A-\varphi_B)]_q^2\}-F_-\{\partial_x(\varphi_A-\varphi_B)\partial_x(\varphi_A+\varphi_B)\}_q\end{pmatrix}. \tag{60}$$

The different terms of both matrices are provided in the following large expressions:

$$D_\pm=q^2+\frac{Qq^2B_\pm(q)[\alpha A_\mp(q)-2k\bar{c}]}{A_-(q)A_+(q)+q^2\bar{c}^2}-\frac{2kq^2[2kA_\pm(q)+\alpha\bar{c}q^2]}{A_-(q)A_+(q)+q^2\bar{c}^2}, \tag{61}$$

$$E_\pm=i\left\{q\bar{c}-\frac{QqB_\pm(q)[2kA_\mp(q)+\alpha\bar{c}q^2]}{A_-(q)A_+(q)+q^2\bar{c}^2}-\frac{2kq^3[\alpha A_\pm(q)-2k\bar{c}]}{A_-(q)A_+(q)+q^2\bar{c}^2}\right\}, \tag{62}$$

$$F_+=\frac{\alpha}{2}+\frac{k\bar{c}q^2}{A_-(q)A_+(q)+q^2\bar{c}^2}-\frac{QA_-(q)B_+(q)}{A_-(q)A_+(q)+q^2\bar{c}^2}, \tag{63}$$

$$F_-=\alpha+\frac{2k\bar{c}q^2}{A_-(q)A_+(q)+q^2\bar{c}^2}-\frac{QA_+(q)B_-(q)}{A_-(q)A_+(q)+q^2\bar{c}^2}, \tag{64}$$

$$G_+=i\left\{\frac{2kqA_+(q)}{A_-(q)A_+(q)+q^2\bar{c}^2}-\frac{\bar{c}QqB_+(q)}{A_-(q)A_+(q)+q^2\bar{c}^2}\right\}, \tag{65}$$

$$G_+=i\left\{\frac{kqA_-(q)}{A_-(q)A_+(q)+q^2\bar{c}^2}-\frac{\bar{c}QqB_-(q)}{2[A_-(q)A_+(q)+q^2\bar{c}^2]}\right\}, \tag{66}$$

and

$$A_\pm(q)=2(1\pm\gamma)Q^2+q^2,$$

$$B_\pm(q)=2(\beta\pm\delta)Q+\frac{\alpha q^2}{Q}, \tag{67}$$

$$\bar{c}=c+2\alpha k.$$

The solutions of the linear part of the phase equation (55) are

$$\begin{pmatrix}(\varphi_A+\varphi_B)_q\\(\varphi_A-\varphi_B)_q\end{pmatrix}=C\exp i\omega_\pm t, \tag{68}$$

where $\omega_\pm$ are the eigenvalues of the linear evaluation matrix $\mathbf{L}^*$, and

$$\omega_\pm=-\frac{\mathrm{Tr}}{2}\pm\left[\left(\frac{\mathrm{Tr}}{2}\right)^2-S\right]^{1/2}, \tag{69}$$

where Tr is the trace $\mathrm{Tr}=D_++D_-$ and S the determinant $S=D_+D_--E_+E_-$.

By computing the trace as an expansion in q up to the fourth order,

$$\frac{\mathrm{Tr}}{2}=\left\{1+\frac{\alpha(\beta-\delta\gamma)}{1-\gamma^2}-\frac{k\overline{c}\beta}{(1-\gamma^2)Q^2}-\frac{2k^2}{(1-\gamma^2)Q^2}\right\}q^2+\left\{\frac{\alpha^2}{2(1-\gamma^2)Q^2}-\frac{\alpha(\beta-\delta\gamma)}{(1-\gamma^2)^2Q^2}+\frac{2k^2}{(1-\gamma^2)^2Q^4}+\frac{\alpha\beta}{2(1-\gamma^2)Q^4}\right.$$
$$\left.-\frac{\alpha k\overline{c}}{(1-\gamma^2)Q^4}-\frac{k^2}{(1-\gamma^2)Q^4}-\frac{\alpha\overline{c}^2(\beta-\delta\gamma)}{4(1-\gamma^2)^2Q^4}+\frac{k\overline{c}\beta}{(1-\gamma^2)Q^4}+\frac{k^2\overline{c}^2}{2(1-\gamma^2)^2Q^6}+\frac{k\overline{c}^3\beta}{4(1-\gamma^2)^2Q^4}\right\}q^4+0(q^6q^8) \tag{70}$$

is obtained. In addition, the determinant S follows the relation

$$S=Aq^2+Bq^4+Cq^6, \tag{71}$$

with the coefficients

$$A=\overline{c}^2+\frac{4k^2(\beta^2-\delta^2)}{1-\gamma^2}-\frac{4k\overline{c}(\beta-\delta\gamma)}{1-\gamma^2}, \tag{72}$$

$$B=1+\frac{2\alpha(\beta-\delta\gamma)}{1-\gamma^2}-\frac{4k^2(\beta-\delta\gamma)}{(1-\gamma^2)Q^2}-\frac{4k\overline{c}\beta}{(1-\gamma^2)Q^2}+\frac{\alpha^2(\beta^2-\delta^2)}{1-\gamma^2}-\frac{2\alpha k^2}{Q^2}\left[\frac{\beta+\delta}{(1-\gamma^2)}+\frac{\beta-\delta}{(1-\gamma^2)}\right]-\frac{2\alpha k\overline{c}}{(1-\gamma^2)Q^2}+\frac{4k^4}{(1-\gamma^2)Q^4}$$
$$+\frac{k^2\overline{c}^2(\beta^2-\delta^2)}{(1-\gamma^2)Q^4}-\frac{4\alpha k\overline{c}}{(1-\gamma^2)Q^2}+\frac{4k\overline{c}(\beta-\delta\gamma)}{(1-\gamma^2)^2Q^2}-\frac{\overline{c}^2\beta}{(1-\gamma^2)Q^2}+\frac{2k^2\overline{c}^2}{(1-\gamma^2)Q^4}+\frac{k\overline{c}^3(\beta-\delta\gamma)}{(1-\gamma^2)^2Q^4}+\frac{8\alpha k^2(\beta-\delta\gamma)}{(1-\gamma^2)^2Q^2}$$
$$-\frac{4k^2(\beta^2-\delta^2)}{(1-\gamma^2)^2Q^2}+\frac{2k\overline{c}(\beta^2-\delta^2)}{(1-\gamma^2)^2Q^2}-\frac{2k^2\overline{c}^2(\beta^2-\delta^2)}{(1-\gamma^2)^2Q^4} \tag{73}$$

and

$$C=\frac{\alpha^2}{(1-\gamma^2)Q^2}-\frac{2\alpha(\beta-\delta\gamma)}{(1-\gamma^2)^2Q^2}+\frac{4k^2}{(1-\gamma^2)^2Q^4}+\frac{\alpha\beta}{(1-\gamma^2)Q^4}-\frac{2\alpha k\overline{c}}{(1-\gamma^2)Q^4}-\frac{2k^2}{(1-\gamma^2)^2Q^4}-\frac{\alpha\overline{c}^2(\beta-\delta\gamma)}{2(1-\gamma^2)^2Q^4}+\frac{2k\overline{c}\beta}{(1-\gamma^2)^2Q^6}$$
$$+\frac{k\overline{c}^3\beta}{(1-\gamma^2)^2Q^6}+\frac{\alpha^3\beta}{(1-\gamma^2)Q^2}-\frac{2\alpha^2(\beta^2-\delta^2)}{(1-\gamma^2)^2Q^2}+\frac{4\alpha k^2}{(1-\gamma^2)Q^4}\left[\frac{\beta+\delta}{(1+\gamma)^2}+\frac{\beta-\delta}{(1+\gamma)^2}\right]+\frac{k\overline{c}^3\beta}{2(1-\gamma^2)^2Q^4}-\frac{2\alpha^2k\overline{c}(\beta-\delta\gamma)}{(1-\gamma^2)^2Q^4}$$
$$+\frac{2\alpha k^2(\beta-\delta\gamma)}{(1-\gamma^2)^2Q^4}-\frac{\alpha^2\overline{c}^2(\beta^2-\delta^2)}{2(1-\gamma^2)^2Q^4}+\frac{2\alpha k\overline{c}(\beta^2-\delta^2)}{(1-\gamma^2)^2Q^4}+\frac{\alpha k^2\overline{c}^2(1-\gamma^2)}{(1-\gamma^2)^3Q^4}+\frac{\alpha k\overline{c}^3(\beta^2-\delta^2)}{(1-\gamma^2)^3Q^6}+\frac{2\alpha^2k^2}{(1-\gamma^2)Q^6}-\frac{4k^4}{(1-\gamma^2)^2Q^6}$$
$$-\frac{2\alpha k^2(\beta-\delta\gamma)}{(1-\gamma^2)Q^6}+\frac{\alpha k^2\overline{c}^2}{2(1-\gamma^2)Q^6}\left[\frac{\beta+\delta}{(1+\gamma)^2}+\frac{\beta-\delta}{(1+\gamma)^2}\right]-\frac{4k^3c(\beta-\delta\gamma)}{2(1-\gamma^2)^2Q^6}-\frac{k^4\overline{c}^2}{(1-\gamma^2)Q^8}-\frac{k^3\overline{c}^3(\beta-\delta\gamma)}{(1-\gamma^2)^3Q^8}-\frac{\alpha^2k\overline{c}(\beta+\delta\gamma)}{2(1-\gamma^2)^2Q^4}$$
$$+\frac{2\alpha k\overline{c}(\beta^2-\delta^2)}{(1-\gamma^2)^3Q^4}-\frac{\alpha k\overline{c}(\beta^2-\delta^2)}{(1-\gamma^2)^2Q^6}+\frac{2\alpha k^2\overline{c}^2\beta}{(1-\gamma^2)^2Q^6}-\frac{2k^2\overline{c}^2(\beta^2-\delta^2)}{(1-\gamma^2)^2Q^6}-\frac{k^2\overline{c}^4(\beta^2-\delta^2)}{4(1-\gamma^2)^3Q^8}-\frac{8\alpha k^2(\beta-\delta\gamma)}{(1-\gamma^2)^3Q^4}+\frac{2\alpha k\overline{c}\beta}{(1-\gamma^2)^2Q^4}$$
$$-\frac{2\alpha k^2\overline{c}^2(\beta-\delta\gamma)}{(1-\gamma^2)^3Q^6}-\frac{2k\overline{c}(\beta^2-\delta^2)}{(1-\gamma^2)^3Q^4}+\frac{k^2c^2(\beta^2-\delta^2)}{(1-\gamma^2)^3Q^6}-\frac{k^2\overline{c}^2\beta}{(1-\gamma^2)^2Q^6}+\frac{k^2(\beta^2-\delta^2)}{(1-\gamma^2)^2Q^4}+\frac{k\overline{c}(\beta^2-\delta^2)}{(1-\gamma^2)^2Q^4}+\frac{\overline{c}^2(\beta^2-\delta^2)}{4(1-\gamma^2)^2Q^4}$$
$$-\frac{k\overline{c}^3(\beta^2-\delta^2)}{2(1-\gamma^2)^3Q^6}. \tag{74}$$

From these expressions, it follows that

$$\omega_{\pm}=-(aq^2+bq^4)\pm i\left\{\sqrt{A}q+\frac{B-a^2}{2\sqrt{A}}q^3\right\}+O(q^5). \tag{75}$$

Hence, at the leading order in q, the real part of ω may be expressed as

$$\mathrm{Re}\,\omega_{\pm}=\left\{1+\frac{\alpha(\beta-\delta\gamma)}{1-\gamma^2}-\frac{k\overline{c}\beta}{(1-\gamma^2)Q^2}-\frac{2k^2}{(1-\gamma^2)Q^2}\right\}q^2, \tag{76}$$

and the standing wave solutions are phase stable for

$$1+\frac{\alpha(\beta-\delta\gamma)}{1-\gamma^2}-\frac{k\overline{c}\beta}{(1-\gamma^2)Q^2}-\frac{2k^2}{(1-\gamma^2)Q^2}\geqslant 0, \tag{77}$$

or

$$Q^2\geqslant\frac{k\overline{c}\beta+2k^2}{(1-\gamma^2)+\alpha(\beta-\delta\gamma)}. \tag{78}$$

In substituting for $Q^2=(\mu-k^2)/(1+\gamma)$, and for $\overline{c}=c+2\alpha k$ in the preceding equation, the following phase stability condition is obtained:

$$\mu \geqslant \frac{k^2[(1+\gamma)(3+2\alpha\beta-\gamma)+\alpha(\beta-\delta\gamma)]+kc\beta(1+\gamma)}{(1-\gamma^2)+\alpha(\beta-\delta\gamma)}. \tag{79}$$

Note that the stability analysis performed in this study is relevant to extended perturbations. For localized perturbations, the results presented here may be affected by the presence of large group velocities. In effect, in this case, the convective and absolute instabilities of the various reference states has to be considered, and the stability domains of the wave patterns may be modified accordingly. From the values of the kinetic coefficients of the corresponding Ginzburg-Landau equations [20], such modifications may be expected to be irrelevant to diluted polymeric fluids, not to Maxwellian fluids, as will be shown in a forthcoming publication [23].

VI. CONCLUSION

In this paper, we presented the derivation of the amplitude equations of the convective patterns that arise in viscoelastic fluids heated from below. Furthermore, we studied the pattern selection and stability as a function of the viscoelastic parameters in the case of Oldroyd-B constitutive equations. At fixed Prandtl number and the ratio between retardation and relaxation times, it may be observed that, on increasing the stress relaxation time, stationary convection may be replaced with oscillatory convection. The analysis of the associated Ginzburg-Landau equation shows that this oscillatory convection should appear in the form of standing waves via a supercritical bifurcation, for wide parameter ranges.

Also, we also derived the phase dynamics and determined the phase stability of the standing waves. Due to the large values of the group velocity, the decay of phase perturbations is usually oscillatory. The phase dynamics appears in the form of two coupled Kuramoto-Sivashinsky equations for the individual phases of the underlying traveling waves. Since we computed the coefficients of these equations at the dominant orders, it would be easy to analyze, in a further study, the defect behavior of the wave pattern.

Our results clearly show that the properties of the bifurcations and of the selected spatiotemporal patterns may change drastically with the exact nature of the constitutive equation used to describe the viscoelastic fluids. Hence, despite the difficulties associated with the possibility of experimental observation of overstable viscoelastic Rayleigh-Bénard instabilities [24], it would be highly desirable to obtain precise experimental data with fluids presenting well-defined viscoelastic properties. Comparisons with the theoretical analysis, such as the one presented here, would then permit us to make a better fitting of the constitutive equations to real fluid properties.

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